JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **163**, 317–321 (1992)

Some Refinements of Ky Fan's Inequality

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Submitted by J. L. Brenner

Received June 1, 1990

1. INTRODUCTION

In 1961 E. F. Beckenbach and R. Bellman published in their well-known book "Inequalities" the following "unpublished result due to Ky Fan" [1, p. 5]:

THEOREM 1. *If* $x_i \in (0, 1/2]$, $i = 1, ..., n$, *then*

$$\left[\prod_{i=1}^{n} x_i \bigg/ \prod_{i=1}^{n} (1 - x_i)\right]^{1/n} \leqslant \sum_{i=1}^{n} x_i \bigg/ \sum_{i=1}^{n} (1 - x_i) \tag{1}$$

with equality only if $x_1 = \cdots = x_n$.

In [3], N. Levinson has obtained an elegant generalization of (1):

THEOREM 2. *Let* $x_i \in (0, a]$ *and* $p_i > 0$, $i = 1, ..., n$, *with* $P_n := \sum_{i=1}^{n} p_i$. *If the function* f *has a nonnegative third derivative on* $(0, 2a)$, *then*

$$\begin{aligned}\frac{1}{P_n} \sum_{i=1}^{n} f(x_i)\, p_i - f\left(\frac{1}{P_n} \sum_{i=1}^{n} x_i p_i\right)& \\ \leqslant \frac{1}{P_n} \sum_{i=1}^{n} f(2a - x_i)\, p_i - f&\left(\frac{1}{P_n} \sum_{i=1}^{n} (2a - x_i)\, p_i\right).\end{aligned} \tag{2}$$

If the third derivative of f *is positive on* $(0, 2a)$, *then equality holds in* (2) *iff* $x_1 = \cdots = x_n$.

If we set $a = 1/2$, $p_1 = \cdots = p_n = 1/n$, and $f = ln$ in (2) then we obtain Fan's inequality (1).

For some inequalities of Levinson's type see [2, 4, 6] where further references are given.

In the following, we shall point out some refinements of Ky Fan's inequality via Levinson's result.

0022-247X/92 $3.00

2. The Main Results

In the recent paper [5], J. E. Pečarić and S. S. Dragomir have obtained the following refinement of Jensen's inequality.

THEOREM 3. *Let X be a real linear space, C be its convex subset, and $f\colon C \to R$ be a convex (concave) function on C. If $x_i \in C$, $p_i \geqslant 0$, $i = 1, ..., n$, $x_{i_j} \in \{x_i\}_{i=1,...,n}$, $p_{j_i} \in \{p_i\}_{i=1,...,n}$, $j = 1, ..., k$, $k = 1, ..., n-1$, and $P_n > 0$, then*

$$
\begin{aligned}
f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) &\leqslant (\geqslant) \frac{1}{P_n^{k+1}} \sum_{i_1,...,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right)\\
&\leqslant (\geqslant) \frac{1}{P_n^{k}} \sum_{i_1,...,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right) \leqslant (\geqslant) \cdots\\
&\leqslant (\geqslant) \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i).
\end{aligned}
$$

Some applications of this fact to the famous arithmetic-geometric inequality are also given.

By the use of this fact we shall point out the following result of Levinson's type.

COROLLARY 3.1. *Let $f\colon [0, A] \to \mathbb{R}$ be a function such that the mapping $g(x) := f(x) - f(A-x)$ is (convex) concave on $[0, \bar{A}]$, $\bar{A} \leqslant A$. If $x_i \in [0, \bar{A}]$, $p_i \geqslant 0$, $i = 1, ..., n$, $x_{i_j} \in \{x_i\}_{i=1,...,n}$, $p_{i_j} \in \{p_i\}_{i=1,...,n}$, and $P_n > 0$, then*

$$
\begin{aligned}
&f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} (A-x_i) p_i\right)\\
&\quad\geqslant (\leqslant) \frac{1}{P_n^{k+1}} \sum_{i_1,...,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} x_{i_j}\right)\\
&\quad\quad - \frac{1}{P_n^{k+1}} \sum_{i_1,...,i_{k+1}=1}^{n} p_{i_1}\cdots p_{i_{k+1}} f\left(\frac{1}{k+1}\sum_{j=1}^{k+1} (A-x_{i_j})\right)\\
&\quad\geqslant (\leqslant) \frac{1}{P_n^{k}} \sum_{i_1,...,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} x_{i_j}\right)\\
&\quad\quad - \frac{1}{P_n^{k}} \sum_{i_1,...,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{k}\sum_{j=1}^{k} (A-x_{i_j})\right)\\
&\quad\geqslant (\leqslant) \cdots \geqslant (\leqslant) \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(A-x_i). \qquad (3)
\end{aligned}
$$

The proof follows from Theorem 3 for the (convex) concave mapping g and we omit the details.

Remark 1. If the inequality (2) is written as

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right)-f\left(\frac{1}{P_n}\sum_{i=1}^{n}(2a-x_i)\,p_i\right)$$
$$\geqslant \frac{1}{P_n}\sum_{i=1}^{n} f(x_i)\,p_i-\frac{1}{P_n}\sum_{i=1}^{n} f(2a-x_i)\,p_i,$$

then inequality (3) may be regarded as a refinement of Levinson's inequality.

Indeed, if f has a nonnegative third derivative on $(0, 2a)$ then $g''(x)=f''(x)-f''(2a-x)\leqslant 0$ because f'' is nondecreasing on $(0, a]$ and $f''(x)\leqslant f''(2a-x)$ for $x\in(0, a]$; i.e., $g(x)=f(x)-f(A-x)$ $(A=2a)$ is concave on $(0, a]$ and the statement is proved.

COROLLARY 3.2. *If* $x_i\in(0, 1/2]$, $i=1, ..., n$, $x_{i_j}\in\{x_i\}_{i=1,...,n}$, $j=1, ..., k$; $k=1, ..., n-1$; *then*

$$\sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n}(1-x_i)\geqslant\left(\prod_{i_1,...,i_{k+1}=1}^{n}\ \sum_{j=1}^{k+1} x_{i_j}\Big/\prod_{i_1,...,i_{k+1}=1}^{n}\ \sum_{j=1}^{k+1}(1-x_{i_j})\right)^{1/n^{k+1}}$$
$$\geqslant\left(\prod_{i_1,...,i_k=1}^{n}\ \sum_{j=1}^{k} x_{i_j}\Big/\prod_{i_1,...,i_k=1}^{n}\ \sum_{j=1}^{k}(1-x_{i_j})\right)^{1/n^{k}}\geqslant\cdots$$
$$\geqslant\left(\prod_{i=1}^{n} x_i\Big/\prod_{i=1}^{n}(1-x_i)\right)^{1/n}.$$

The equality holds in all inequalities iff $x_1=\cdots=x_n$.

The proof is obvious from Corollary 3.1 for $f=ln$, $p_1=\cdots=p_n=1/n$, $A=1$, and $\bar{A}=1/2$.

Now, we shall establish another refinement of Jensen's inequality which will be used to obtain some new improvements of Levinson's and Fan's inequalities.

THEOREM 4. *Let f be a convex (concave) real function defined on a convex set C $(C\subseteq X)$, $x_i\in C$, $p_i\geqslant 0$, $i=1, ..., n$, $P_n>0$, and $x_{i_j}\in\{x_i\}_{i=1,...,n}$,*

$p_{i_j} \in \{p_i\}_{i=1,\dots,n}$ for $j=1,\dots,k$; $k=1,\dots,n$. Then for all $q_j \geqslant 0$ such that $Q_k > 0$ we have

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant (\geqslant) \frac{1}{P_n^k} \sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right)$$
$$\leqslant (\geqslant) \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{4}$$

Proof. Let $1 \leqslant k \leqslant n$. Then, by Jensen's inequality, we have

$$f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right) \leqslant (\geqslant) \frac{1}{Q_k}\sum_{j=1}^{k} q_j f(x_{i_j}),$$

for all $i_1, \dots, i_k \in \{1, 2, \dots, n\}$.

Multiplying these inequalities with $p_{i_1}\cdots p_{i_k} \geqslant 0$ and summing after $i_1, \dots, i_k$ to 1 at n, we derive

$$\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right)$$
$$\leqslant (\geqslant) \frac{1}{Q_k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\sum_{j=1}^{k} q_j f(x_{i_j})\right).$$

Since a simple calculus shows that

$$\frac{1}{Q_k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\sum_{j=1}^{k} q_j f(x_{i_j})\right) = P_n^{k-1}\sum_{i=1}^{n} p_i f(x_i),$$

then the above inequality yields the second part of (4).

To prove the first part of (4) we need the inequality

$$\frac{1}{P_n^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k} f\left(\frac{1}{Q_k}\sum_{j=1}^{k} q_j x_{i_j}\right)$$
$$\geqslant (\leqslant) f\left(\frac{1}{Q_k P_n^k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j}\right)\right),$$

which holds from Jensen's inequality for k-variables.

Since

$$\frac{1}{Q_k}\sum_{i_1,\dots,i_k=1}^{n} p_{i_1}\cdots p_{i_k}\left(\sum_{j=1}^{k} q_j x_{i_j}\right) = P_n^{k-1}\sum_{i=1}^{n} p_i x_i,$$

the proof is finished.

COROLLARY 4.1. *Let f, A, $\bar{A}$, x_i, p_i, $i=1, \ldots, n$, be as in Corollary* 3.1. *Then for all $q_j \geqslant 0$, $j=1, \ldots, k$, with $Q_k > 0$, $k=1, \ldots, n$, we have*

$$\begin{aligned} & f\left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i\right) - f\left(\frac{1}{P_n} \sum_{i=1}^{n} (A - x_i) p_i\right) \\ & \geqslant (\leqslant) \frac{1}{P_n^k} \sum_{i_1, \ldots, i_k = 1}^{n} p_{i_1} \cdots p_{i_k} f\left(\frac{1}{Q_k} \sum_{j=1}^{k} q_j x_{i_j}\right) \\ & \quad - \frac{1}{P_n^k} \sum_{i_1, \ldots, i_k = 1}^{n} p_{i_1} \cdots p_{i_k} f\left(\frac{1}{Q_k} \sum_{j=1}^{k} q_j (A - x_{i_j})\right) \\ & \geqslant (\leqslant) \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(A - x_i). \end{aligned}$$

Finally, we have another refinement of Fan's inequality.

COROLLARY 4.2. *Let $x_i \in (0, 1/2]$, $i=1, \ldots, n$; $x_{i_j} \in \{x_i\}_{i=1, \ldots, n}$ and $q_j \geqslant 0$ such that $Q_k > 0$, $j=1, \ldots, k$; $k=1, \ldots, n$. Then we have*

$$\begin{aligned} \sum_{i=1}^{n} x_i \Big/ \sum_{i=1}^{n} (1 - x_i) & \geqslant \left(\prod_{i_1, \ldots, i_k = 1}^{n} \sum_{j=1}^{k} q_j x_{i_j} \Big/ \prod_{i_1, \ldots, i_k = 1}^{n} \sum_{j=1}^{k} q_j (1 - x_{i_j}) \right)^{1/n^k} \\ & \geqslant \left(\prod_{i=1}^{n} x_i \Big/ \prod_{i=1}^{n} (1 - x_i) \right)^{1/n}. \end{aligned}$$

The equality holds in all inequalities iff $x_1 = \cdots = x_n$.

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